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РЕЗОНАНСНЫЕ СОСТОЯНИЯ РЕЛЯТИВИСТСКИХ СИСТЕМ И КОВАРИАНТНЫЕ ДВУХЧАСТИЧНЫЕ УРАВНЕНИЯ

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RESONANCE STATES OF RELATIVISTIC SYSTEMS AND COVARIANT TWO-PARTICLE EQUATIONS

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В работе описан метод, позволяющий находить резонансные состояния для релятивистских двухчастичных систем и исследовать их влияние на сечение рассеяния. Метод основан на решении интегральных уравнений в релятивистском конфигурационном представлении. Данный метод применён для идентификации структур сечения рассеяния для модельного потенциала.

Ключевые слова: *двухчастичные интегральные уравнения, релятивистское конфигурационное представление, комплексный поворот, резонансное состояние, амплитуда рассеяния, сечение рассеяния.*

Method for determination of resonance states of the relativistic two-particle system and analysis of their influence on the cross section is presented. The method is based on the integral equations in the relativistic configurational representation. This method is applied for the identification of the scattering cross sections structures for a model potential.

Keywords: *two-particle integral equations, relativistic configurational representation, complex scaling, resonance states, scattering amplitude, cross section.*

Introduction

One of the quantum field theory approaches used to describe two-particle systems is based on two-particle equations of the quasipotential type [1], [2]. Initially these equations were obtained in the momentum representation where they have integral form analogous to the Schrodinger integral equation. Alternatively, the so called relativistic configurational representation (RCR) [3], [4] for the two-particle quasipotential equations is commonly used. The RCR is introduced by means of expansion of all values in the equations over matrix elements of the principal series of the Lorentz group irreducible unitary representations [5]. One of the advantages of the RCR equations in comparison with the equations in the momentum representation is the physical sense transparency of the potentials. For example the analytic dependence of the potential $V(r)$ on the RCR-variable r may indicate the probable presence of resonant or bound states for the considered system. In this paper we describe model-potential study of the RCR integral equations for the resonance states of the two particle systems.

1 Complex scaling for the relativistic integral equations

Two-particle integral equations for the scattering s-states in the RCR have the following form [6]

$$\psi_{(j)}(\chi_q, r) = \sin\chi_q mr + \int_0^\infty G_{(j)}(\chi_q, r, r')V(r')\psi_{(j)}(\chi_q, r')dr'. \quad (1.1)$$

Here $j = 1$ ($j = 3$) corresponds to the Logunov-Tavkhelidze equation (modified equation), $j = 2$ ($j = 4$) corresponds to the Kadyshevsky equation (modified equation), $\psi_{(j)}(\chi_q, r)$ – wave function, $V(r)$ – relativistic potential, $G_{(j)}(\chi_q, r, r')$ – Green function (GF). The rapidity χ_q is connected to the relativistic energy $2E_q$ through $2E_q = 2m\cosh\chi_q$. Green functions for the specific j are written as [6]

$$G_{(j)}(\chi_q, r, r') = G_{(j)}(\chi_q, r - r') - G_{(j)}(\chi_q, r + r'), \quad (1.2)$$

where

$$G_{(1)}(\chi_q, r) = \frac{-i}{K_q^{(1)}}\frac{\sinh[(\pi/2 + i\chi_q)mr]}{\sinh[\pi mr/2]};$$

$$G_{(3)}(\chi_q, r) = \frac{-i}{K_q^{(3)}}\frac{\cosh[(\pi/2 + i\chi_q)mr]}{\cosh[\pi mr/2]};$$

$$G_{(2)}(\chi_q, r) = \frac{(4m\cosh\chi_q)^{-1}}{\cosh[\pi mr/2]} - \frac{i}{K_q^{(2)}}\frac{\sinh[(\pi + i\chi_q)mr]}{\sinh[\pi mr]};$$

$$G_{(4)}(\chi_q, r) = \frac{-i}{K_q^{(4)}}\frac{\sinh[(\pi + i\chi_q)mr]}{\sinh[\pi mr]}.$$

Here we use the denotations

$$K_q^{(1)} = K_q^{(2)} = m\sinh 2\chi_q,$$
$$K_q^{(3)} = K_q^{(4)} = 2m\sinh\chi_q.$$

In quantum mechanics the resonance states are defined as the *S*-matrix (or scattering amplitude) poles located in the fourth quadrant of the complex momentum *p* plane [7]. In this work we investigate the existence of such poles in the complex rapidity χ_q plane. Relativistic integral equations for the resonance states have to be homogeneous by analogy with non-relativistic case. In these equations GF for the states with complex energy $2E_q = 2\sqrt{q^2+m^2}$ (*m* – mass of each particle) has to be used, where real part of the energy is greater than the rest energy 2*m* of two considered particles.

The integral equations for the resonance states, i.e. states with complex rapidities $\chi_q = \xi_q + iw_q$, where ξ_q, w_q are real parameters have the form

$$\psi_{(j)}(\xi_q + iw_q, r) = \\ = \int_0^\infty dr' G_{(j)}(\xi_q + iw_q, r, r') V(r') \psi_{(j)}(\xi_q + iw_q, r'). \quad (1.3)$$

One can solve integral equations (1.3) numerically only for sufficiently fast decreasing analytical potentials, because GF and wave function in integrals generally do not decrease at $r \to \infty$. Moreover numerical solution of this equations is possible only in the band $w_{min} \le w_q \le w_{max}$, which is dependent on the properties of the potential. However, resonant rapidities may be found outside this area. In order to solve (1.3) in other domain of complex χ_q, we will use the well known non-relativistic complex scaling method [8], [9]. After transformation of the real variables *r, r'* to the complex variables $z = r\exp(i\theta)$, $z' = r'\exp(i\theta)$, $0 \le \theta \le \theta_{max}$ the equations for the resonance states are expressed as

$$\psi^{(\theta)}_{(j)}(\xi_q + iw_q, r) = \quad (1.4) \\ = \int_0^\infty dr' G^{(\theta)}_{(j)}(\xi_q + iw_q, r, r') V^{(\theta)}(r') \psi^{(\theta)}_{(j)}(\xi_q + iw_q, r'),$$

where

$$\psi^{(\theta)}_{(j)}(\chi_q, r) = \psi_{(j)}(\chi_q, z),\ V^{(\theta)}(r') = \\ = \exp(i\theta) V(z'),\ G^{(\theta)}_{(j)}(\chi_q, r, r') = G_{(j)}(\chi_q, z, z').$$

In the non-relativistic theory resonance states correspond to sudden changes of the scattering amplitude in some resonance energy real part neighborhood. Scattering amplitude in the quantum mechanics is defined as the coefficient divided by momentum in front of the scattered wave exp(*ipr*), when asymptotic form of the wave function at $r \to \infty$ is considered. Let us consider the analogous asymptotes of the relativistic wave function. Taking into account asymptotic behavior of the GFs

$$G_{(j)}(\chi_q, r, r')\Big|_{r\to\infty} \cong -\frac{2}{K_q^{(j)}} \sin \chi_q m r' \exp(i\chi_q m r)$$

integral equation (1.1) at $r \to \infty$ yields:

$$\psi_{(j)}(r)\Big|_{r\to\infty} \cong \sin \chi_q m r + q f_{(j)}(\chi_q) \exp(i\chi_q m r),$$

where $q = m\sinh\chi_q$ is the relativistic momentum and scattering amplitude $f_{(j)}(\chi_q)$ has the following form

$$f_{(j)}(\chi_q) = \\ = -\frac{2}{qK_q^{(j)}} \int_0^\infty dr' \sin \chi_q m r' \, V(r') \psi_{(j)}(\chi_q, r'). \quad (1.5)$$

Partial scattering amplitude is connected to the partial *s*-wave cross section $\sigma_{(j)}(\chi_q)$

$$\sigma_{(j)}(\chi_q) = 4\pi |f_{(j)}(\chi_q)|^2. \quad (1.6)$$

2 Resonance contributions into the scattering amplitude

In order to study the influence of the resonances on the scattering amplitude (or the cross section) let us define the contribution of the *R*-th resonance to the scattering amplitude by analogy with the non-relativistic case [10]

$$\frac{\mathrm{Res}[f_{(j)}(\chi_q^R)]}{\chi_q - \chi_q^R}. \quad (2.1)$$

The residue $\mathrm{Res}[f_{(j)}(\chi_q^R)]$ can be found using the Cauchy's theorem [11]

$$\mathrm{Res}[f_{(j)}(\chi_q^R)] = \frac{1}{2\pi i} \oint_C f_{(j)}(\chi_q) d\chi_q = \\ = \frac{\rho}{2\pi} \int_0^{2\pi} f_{(j)}(\chi_q^R + \rho e^{i\phi}) e^{i\phi} d\phi. \quad (2.2)$$

Contour *C* is a circle in the complex χ_q plane with the center at resonant rapidity χ_q^R, ρ is the radius of this circle, chosen in such a way that inside the contour only one resonance of interest is located. Defining the reduced scattering amplitude $\tilde{f}_{(j)}$ and reduced cross section $\tilde{\sigma}_{(j)}$

$$\tilde{f}_{(j)}(\chi_q, \chi_q^R) = f_{(j)}(\chi_q) - \frac{Res[f_{(j)}(\chi_q^R)]}{\chi_q - \chi_q^R}, \\ \tilde{\sigma}_{(j)}\left(\chi_q, \chi_q^R\right) = 4\pi \left|\tilde{f}_{(j)}(\chi_q, \chi_q^R)\right|^2, \quad (2.3)$$

and then comparing $\sigma_{(j)}$ and $\tilde{\sigma}_{(j)}$ one can identify possible feature in the cross section and investigate the influence of the desired resonance.

For the numerical treatment we approximate integrals in (1.1) and (1.4) by one of the quadrature formulas $\int_a^b F(r)dr \cong \sum_{k=1}^N \omega_k F(r_k)$, where ω_k, r_k – are weights and grid points. After approximation we obtain the systems of *N* linear algebraic equations with respect to $\psi_k = \psi_{(j)}(\chi_q, r_k)$ and $\psi_k^{(\theta)} = \psi_{(j)}^{(\theta)}(\xi_q + iw_q, r_k)$, correspondingly:

$$\sum_{k=1}^N M_{sk} \psi_k = b_s, \quad (2.4)$$

$$M_{sk} = \delta_{sk} - \omega_k G_{(j)}(\chi_q, r_s, r_k) V(r_k),\ b_s = \sin\chi_q m r_s,$$

$$\sum_{k=1}^{N} M_{sk}^{(\theta)} \psi_k^{(\theta)} = 0,$$

$$M_{sk}^{(\theta)} = \delta_{sk} - \omega_k G_{(j)}^{(\theta)}(\chi_q, r_s, r_k) V^{(\theta)}(r_k),$$

where δ_{sk} are elements of the identity matrix. Nontrivial solution of the homogeneous system exists if the following condition holds:

$$d_{(j)}(\chi_q) = d_{(j)}(\xi_q + i w_q) = \det(M_{(j)}^{(\theta)}) = 0. \quad (2.5)$$

This condition is satisfied only for some complex χ_q values, which are the rapidities corresponding to the resonance states. Separating real and imaginary parts of $d_{(j)}(\chi_q)$ we rewrite equation (2.5) as the system of nonlinear equations

$$\mathrm{Re}[d_{(j)}(\chi_q)] = 0, \quad \mathrm{Im}[d_{(j)}(\chi_q)] = 0. \quad (2.6)$$

Computing $d_{(j)}(\chi_q)$ for different rapidity values (on the grid in the complex χ_q plane) we find the approximate position of the determinant zeroes, and then use these values as start values for solving system (2.6) by the continuous analog of the Newton's method [12]. Solving non-homogeneous system (2.4) and using approximation of the integrals we are able to compute scattering amplitudes and cross sections and obtain residues of the scattering amplitudes at resonant rapidities.

3 Results of the calculations

In tables 3.1 and 3.2 we present the results of the calculations of the resonant rapidities and scattering amplitude residues for the following potentials

$$V_1(r) = 30r^2 \frac{\cosh(\pi-\beta)mr}{\cosh\pi mr}, \quad V_2(r) = 30r^2 \frac{\sinh(\pi-\beta)mr}{\sinh\pi mr}, \quad (3.1)$$

where $\beta < \pi$ and $m = 1$. These potentials are possible relativistic generalization of the well known non-relativistic potential $V_0 r^2 \exp(-\alpha r)$.

In figure 3.1 cross sections, scattering amplitudes, reduced cross sections, reduced scattering amplitudes and contributions of the resonances to the scattering amplitudes for the potential V_1 with $j = 2$, $\beta = \pi/4$ are presented. The first three plots correspond to the case when contribution from the first resonance is excluded, the second and third plots correspond to the pictures obtained when contributions from the second and third resonances are omitted. For the considered potential the first and the second resonances lie very close to the real axis (see table 3.1). These structures completely disappear for the reduced cross sections and amplitudes. In the figures for the cross section and for the scattering amplitude one can see narrow peak and narrow trough at the corresponding rapidity values. The third resonance has lager imaginary part and has influence on the wider area of the cross section. In figure 3.1 one can see that the contribution of this resonance to the scattering amplitude is much more delocalized. The wide trough disappeare in the reduced cross section, but it is reasonable that structures assigned to the narrow resonances are still visible.

Table 3.1 – Resonant rapidities for the potentials (3.1)

β	R	j = 2, potential V_1		j = 4, potential V_2	
		$\mathrm{Re}[\chi_q]$	$\mathrm{Im}[\chi_q]$	$\mathrm{Re}[\chi_q]$	$\mathrm{Im}[\chi_q]$
$\frac{\pi}{4}$	1 2 3	2.399071 2.767267 2.569008	–0.001305 –0.007430 –0.192597	2.801659 3.339097 3.425283	-4.2×10^{-11} –0.013728 –0.187849
		j = 1, potential V_1		j = 3, potential V_2	
$\frac{\pi}{2}$	1	1.720483	–0.134337	1.868578	–0.257724

Table 3.2 – Scattering amplitude residues for the potentials (3.1)

β	R	j = 2, potential V_1		j = 4, potential V_2	
		$\mathrm{Re}[\chi_q]$	$\mathrm{Im}[\chi_q]$	$\mathrm{Re}[\chi_q]$	$\mathrm{Im}[\chi_q]$
$\frac{\pi}{4}$	1 2 3	-1.9698×10^{-4} 8.6100×10^{-4} -7.5494×10^{-3}	1.3114×10^{-4} 4.9217×10^{-4} 6.1202×10^{-2}	– 6.6597×10^{-4} 9.8482×10^{-2}	– 1.0657×10^{-3} -3.4470×10^{-2}
		j = 1, potential V_1		j = 3, potential V_2	
$\frac{\pi}{2}$	1	-3.4617×10^{-2}	5.1932×10^{-2}	7.4375×10^{-2}	6.4391×10^{-2}

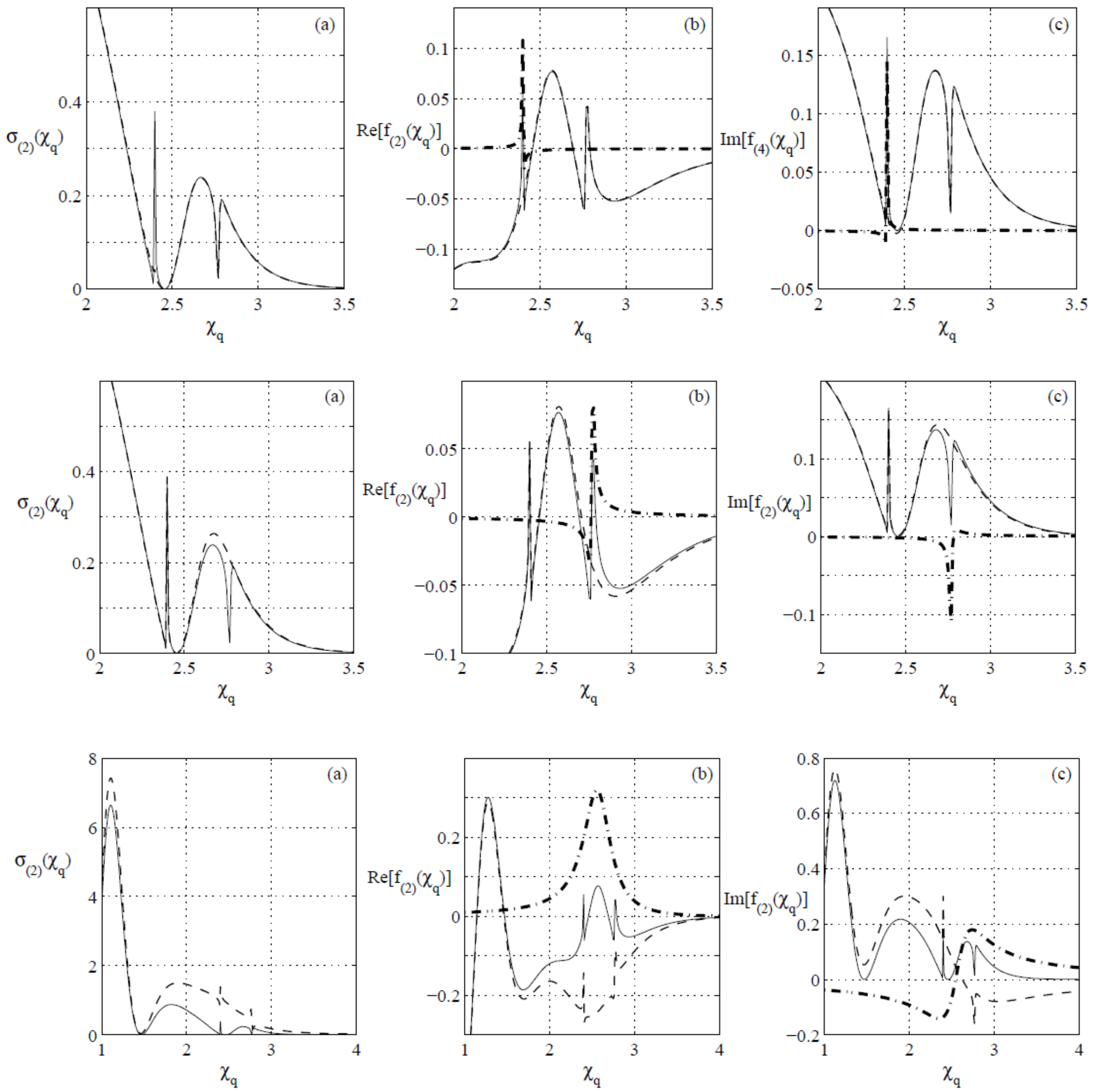

Figure 3.1 – Cross sections and scattering amplitudes
—— – cross section/scattering amplitude; – – – – reduced cross section/reduced scattering amplitude;
– – – – contribution of the resonance into the scattering amplitude

Conclusion

The complex scaling method, widely used in the non-relativistic theory and applied to the RCR two particle integral equations allows the calculation of resonant rapidities for the analytical potentials. Defining the contribution of resonance into the scattering amplitude through its residue gives the possibility to assign the structure in the cross section to the particular resonance. It is shown that analysing the influence of the resonance on the cross section with the use of the presented method it is possible to distinguish the contributions from the overlapping resonance structures. The proposed method may be applied for the study of more realistic systems in the presence of resonance states.

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